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February 12, 1973

NIC #14212
Packet Radio Temporary Note #24

THROUGHPUT IN CARRIER-SENSE (AUTOSLOT) PACKET RADIO SYSTEMS

I. Introduction

In packet-radio systems, the distance between two devices is relatively short so that the propagation delay is small compared to the packet transmission time. For example, if the propagation delay τ is 54 µsec (distance of 10 miles), and the packet transmission time T is 5 msec (500 bits @ 100 Kb/s), then $\tau/T = .011$. This is favorable to the carrier-sense operating mode of the channel which consists of sensing (listening to) the channel before transmission of a packet and acting as follows:

- if the channel is idle, the device transmits the packet.
- if the channel is busy, the device will wait until the channel goes idle, and only then will it transmit the packet.

In this note we analyze the utilization of the channel operating under this mode which proves to be superior to both the pure Aloha and the slotted Aloha modes.

II. Assumptions

1. All packets are assumed to be of the same size.
2. The propagation delay between any pair of devices is assumed to be identical.
3. A terminal can instantaneously switch from the receiving mode to the transmitting mode.
4. The interarrival times of the point process defined by the arrival times of all packets (new and retransmissions) are independent and exponential. (This implicitly assumes that the retransmission delay is large compared to the packet transmission time.)

III. Notation

T : Transmission time of a packet.

τ : Propagation delay between any pair of devices.

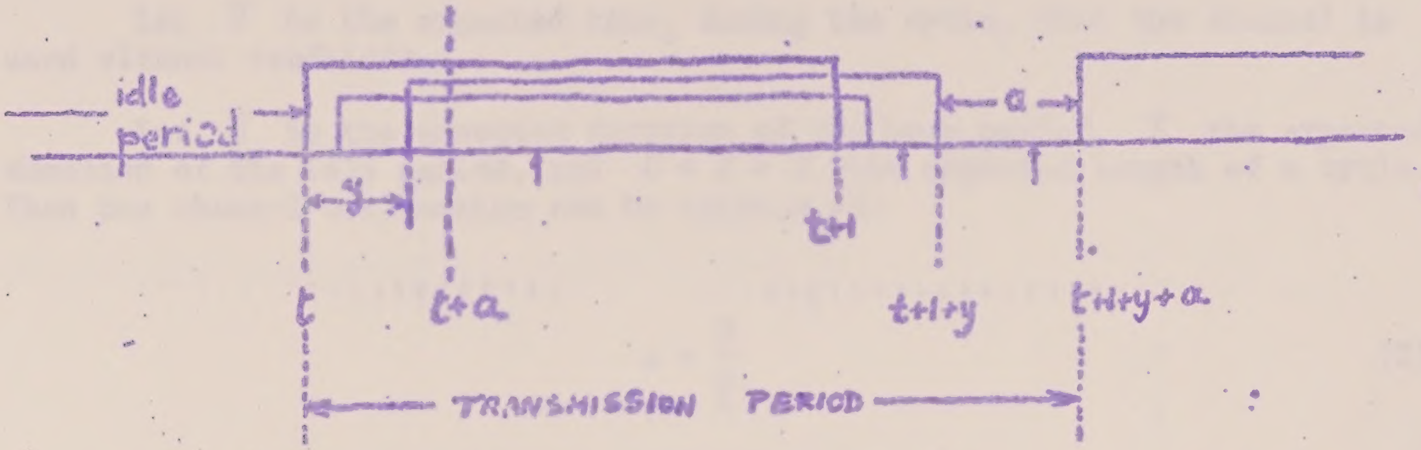
G : Average number of (new) message packets and retransmissions per transmission time. This is the channel traffic.

S : Average generation rate of (new) message packets per transmission time. This is the channel utilization (throughput).

$$a = \frac{\tau}{T}$$

Without loss of generality we take $T = 1$ and $\tau = a$. This is equivalent to expressing time in units of size T.

IV. Channel Utilization



Let t be the time of arrival of a packet that finds the channel idle. Any packet arriving between t and $t + a$ will find (sense) the channel as unused, will transmit and hence will cause a conflict. Let $t + y$ be the time of occurrence of the last packet arriving between t and $t + a$. The transmission of all packets arriving in $(t, t + y)$ will be completed at $t + y + 1$. Only a sec later will the channel be sensed unused.

Now any packet arriving between $t + a$ and $t + y + 1 + a$ will sense the channel busy and hence will wait until the channel is sensed idle $(t + 1 + y + a)$ at which time they will all transmit.

The interval between t and $t + y + 1 + a$ is called a transmission period, and the number of packets accumulated at the end of a transmission period is the number of arrivals in $1 + y$ sec. Note that there can be at most one successful transmission during a transmission period.

Define a busy period to be the time between t and the end of that last transmission period during which no packets accumulate.

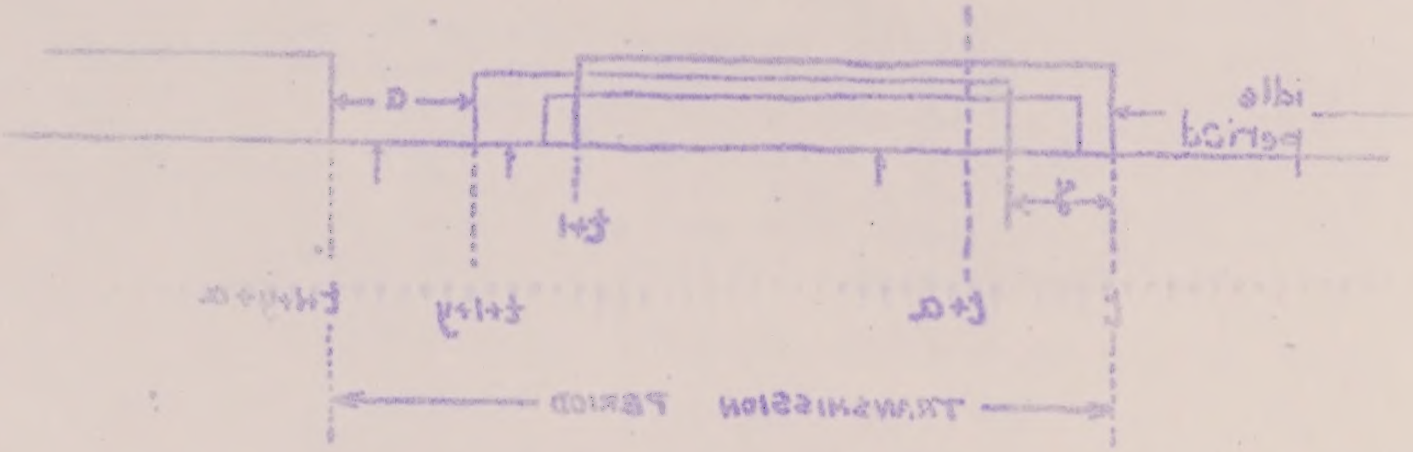
Define an idle period to be the period of time over which the channel is idle and no packets are present. A busy period immediately followed by an idle period constitute a cycle.

Assume $y = a$ for the present time. This, of course, consists of the worst case and will provide us with a lower bound on the utilization. The reader is referred to the Appendix for the exact analysis of the distribution of y. Let $\{q_k\}$ be the distribution of the number of packets accumulated at the end of a transmission period:

$$\frac{\tau}{T} = a$$

Without loss of generality we take $T = 1$ and $\tau = a$. This is equivalent to expressing time in units of size T.

IV. Channel Utilization



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$$q_k = \Pr\{\text{number of packets} = k\} \tag{1}$$

This distribution is Poisson with mean $G(1 + a)$. There are two ways of analyzing the channel utilization: a cycle analysis and a "probability of success" argument, both of which we present below.

IV.1. Cycle Analysis

Let $\bar{U}$ be the expected time, during the cycle, that the channel is used without conflicts.

Let $\bar{B}$ be the expected duration of the busy period, $\bar{I}$ the expected duration of the idle period, and $\bar{C} = \bar{B} + \bar{I}$ the expected length of a cycle. Then the channel utilization can be written as:

$$s = \frac{\bar{U}}{\bar{C}} \tag{2}$$

It is clear that

$$\bar{I} = 1/G \tag{3}$$

It is also clear that the number of transmission periods in a busy period is equal to k with probability $q_0(1 - q_0)^{k-1}$ so that

$$\bar{B} = \sum_{k=1}^{\infty} k(1 + 2a)\, q_0(1 - q_0)^{k-1} = \frac{1 + 2a}{q_0} \tag{4}$$

where

$$q_0 = e^{-G(1 + a)} \tag{5}$$

Let us now dissect a busy period. The probability of success in the first transmission period is e^{-aG}, the probability that no packets arrive during

its first a sec. The probability of success in each subsequent transmission period is $e^{-aG}\frac{q_1}{1-q_0}$. Therefore

$$\overline{U} = \sum_{k=1}^{\infty}\left[e^{-aG} + (k-1)e^{-aG}\frac{q_1}{1-q_0}\right] q_0(1-q_0)^{k-1}$$

$$= e^{-aG}\left(1 + \frac{q_1}{q_0}\right) \qquad (6)$$

The channel utilization is therefore given by

$$S = \frac{\overline{U}}{\overline{B}+\overline{I}} = \frac{Ge^{-G(1+2a)}(1+G+aG)}{G+2aG+e^{-G(1+a)}} \qquad (7)$$

IV.2. Probability of Success

Consider the transmission of an arbitrary packet. Three situations must be considered:

a. If, upon arrival, that packet found the system idle, then its probability of success is e^{-aG} .

b. If, upon arrival, that packet fell in the first a seconds of a transmission period, then the probability of success is 0 .

c. If, upon arrival, that packet found the channel in a busy period excluding the first a sec of the transmission period, then the probability of success is $e^{-G(1+2a)}$.

The probability that an arrival finds the channel idle is given by $\frac{\overline{I}}{\overline{B}+\overline{I}}$, and the probability that an arrival finds the channel in situation (c) is given by $\frac{(1+a)/q_0}{\overline{B}+\overline{I}}$. Therefore the probability of success is

$$\frac{s}{G} = \frac{\frac{(1+a)}{q_0} e^{-G(1+2a)} + \frac{1}{G} e^{-aG}}{\frac{1+2a}{q_0} + \frac{1}{G}}$$

$$= \frac{e^{-G(1+2a)}(1+G+aG)}{G(1+2a) + e^{-G(1+a)}} \qquad (8)$$

which corresponds to our result in Eq. (7) using the cycle analysis.

In the ideal case $(a = 0)$ we have the ultimate performance

$$\frac{s}{G} = \frac{e^{-G}(1+G)}{G+e^{-G}} \qquad (9)$$

Comparing this ideal case to a slotted system for which $s/G = e^{-G}$ in the limit, we see that $(s/G)_{AUTOSLOT} > (s/G)_{SLOTTED}$.

The exact analysis in the Appendix gives

$$\frac{s}{G} = \frac{(1+aG)[G+aG+e^{-aG}]e^{-G(1+2a)}}{G+2aG-(1-e^{-aG})+(1+aG)e^{-G(1+a)}} \qquad (10)$$

Numerical Results

In Fig. 1 we plot the throughput s vs. channel traffic G, using Eq. (10) obtained from the exact analysis (see Appendix).

It is interesting to see that the maximum utilization is very sensitive to a, and that as a increases, the optimum value of s is obtained for decreasing values of G.

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In Fig. 2, we plot the maximum throughput as a function of a (the envelope of all constant G contours) and show that beyond a = 0.225 Slotted Aloha gives better performance. Recall, however, that for practical values we expect a << 1 , and so this autoslot mode appears quite promising.

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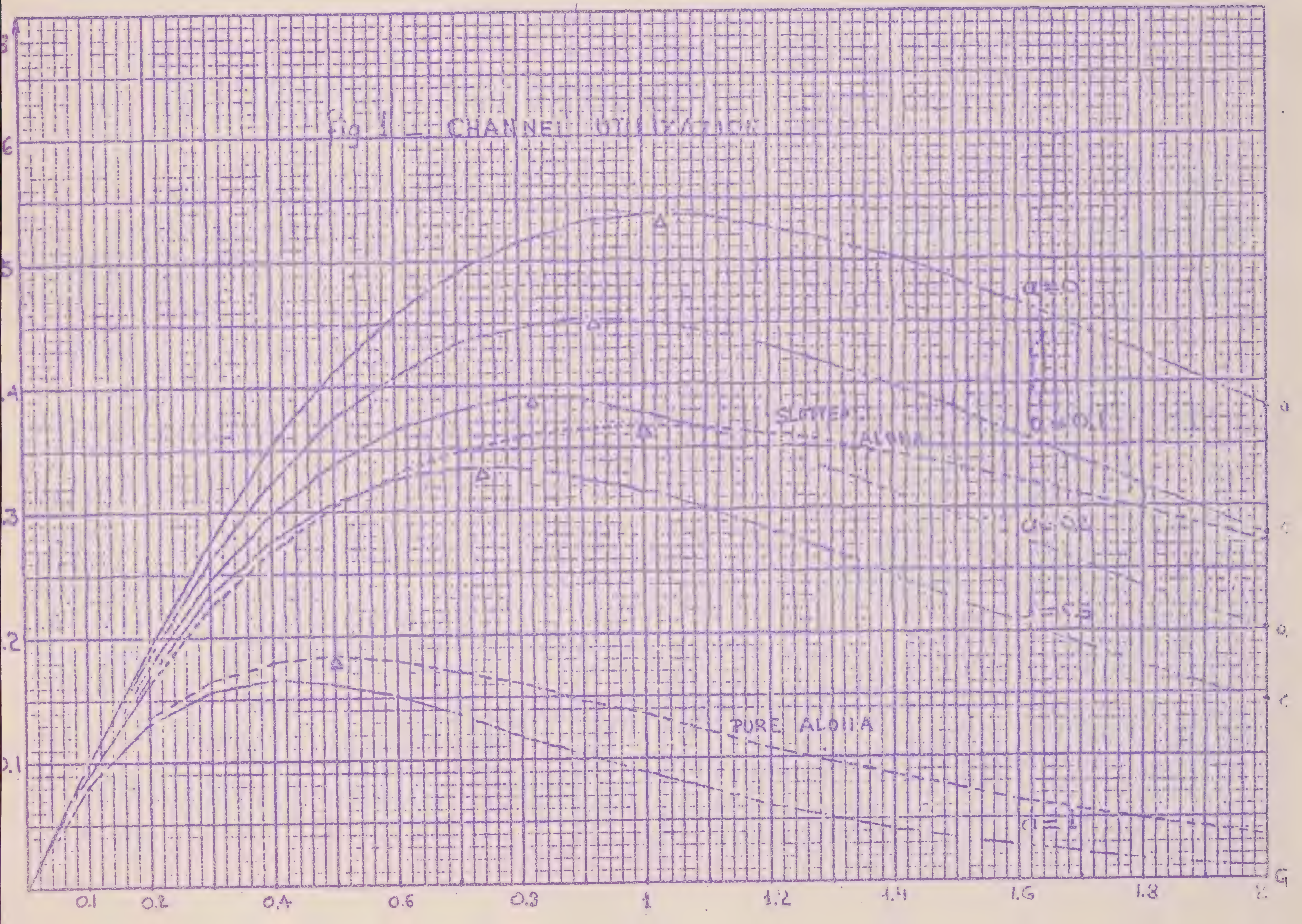

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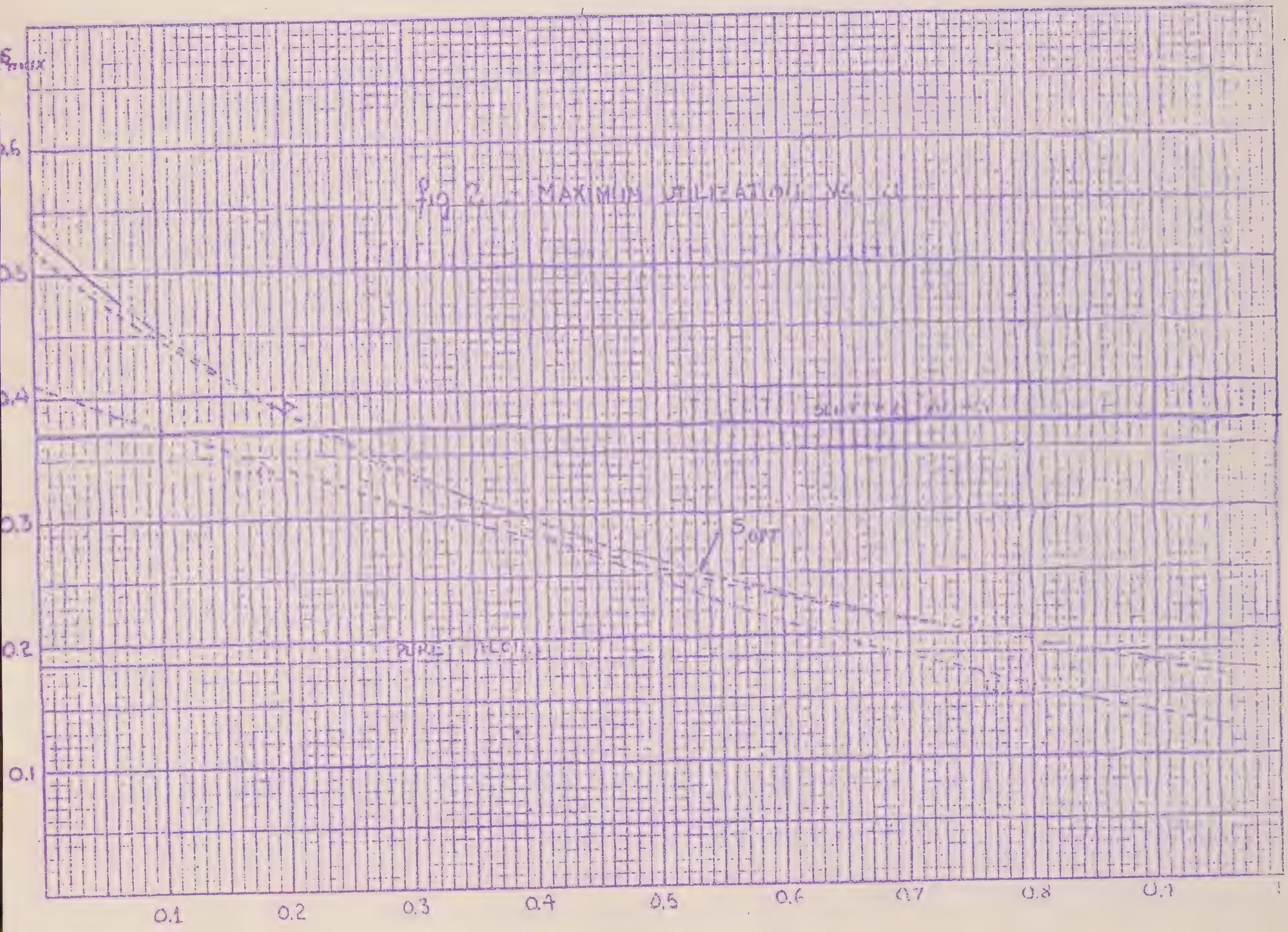

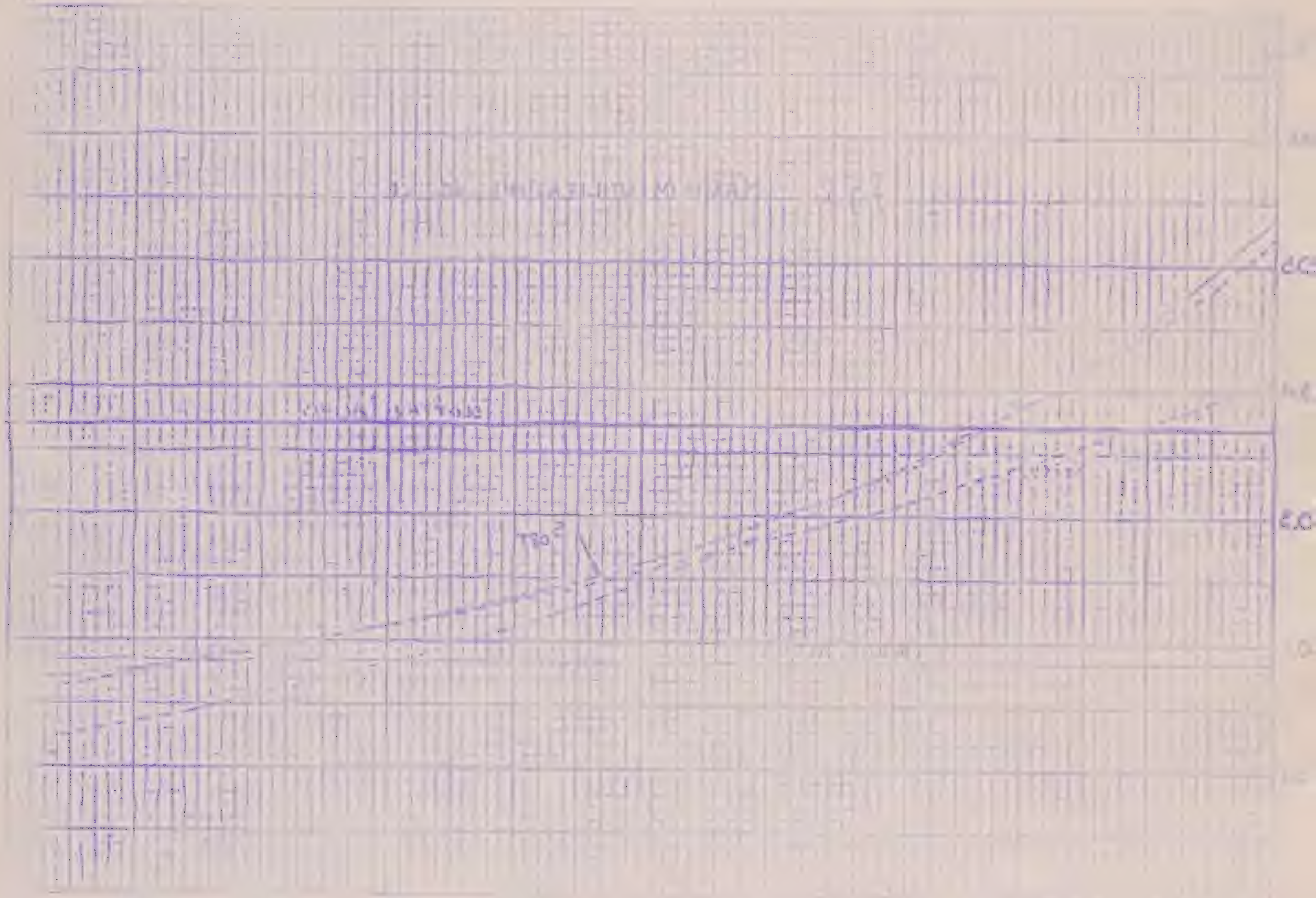

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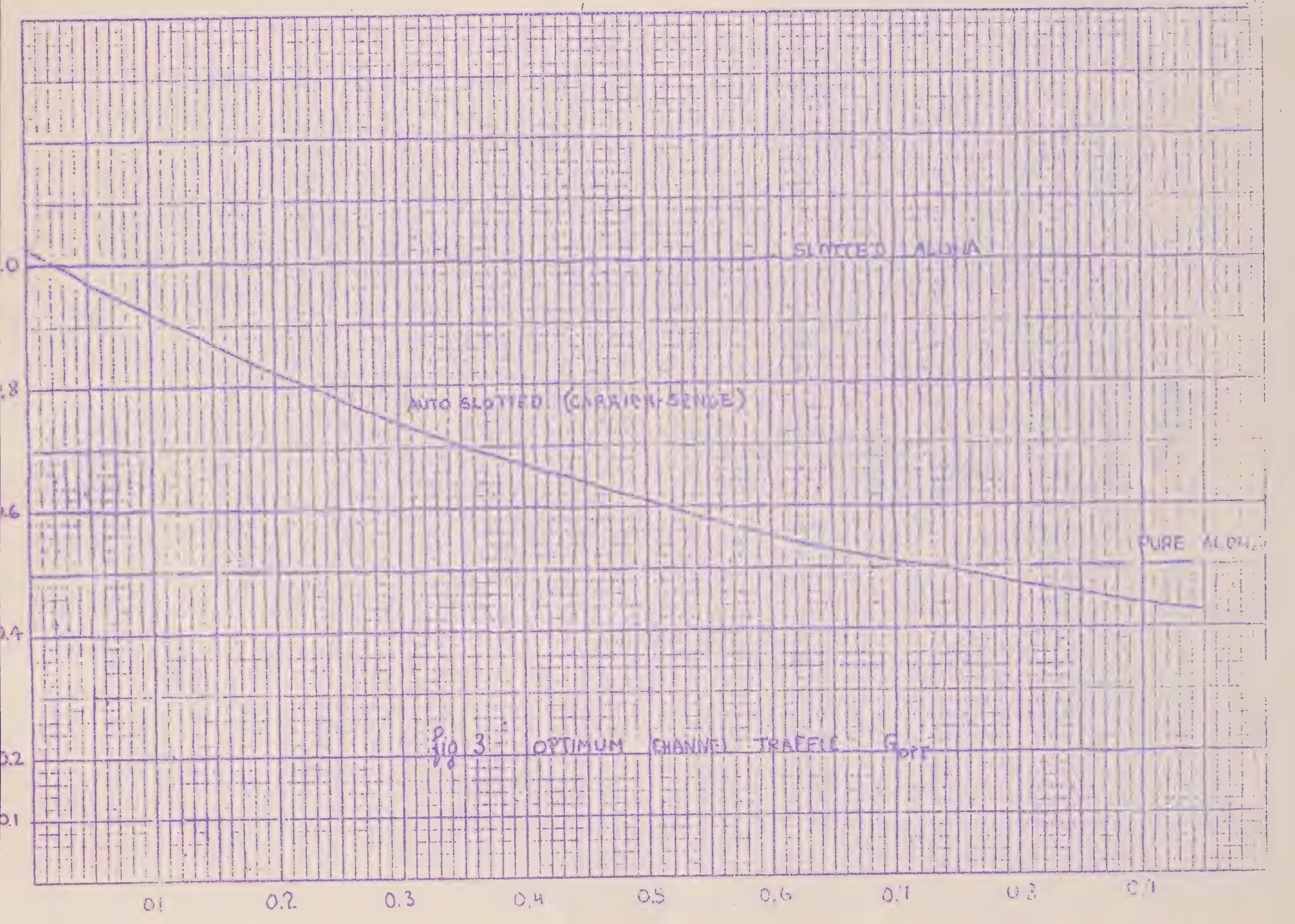
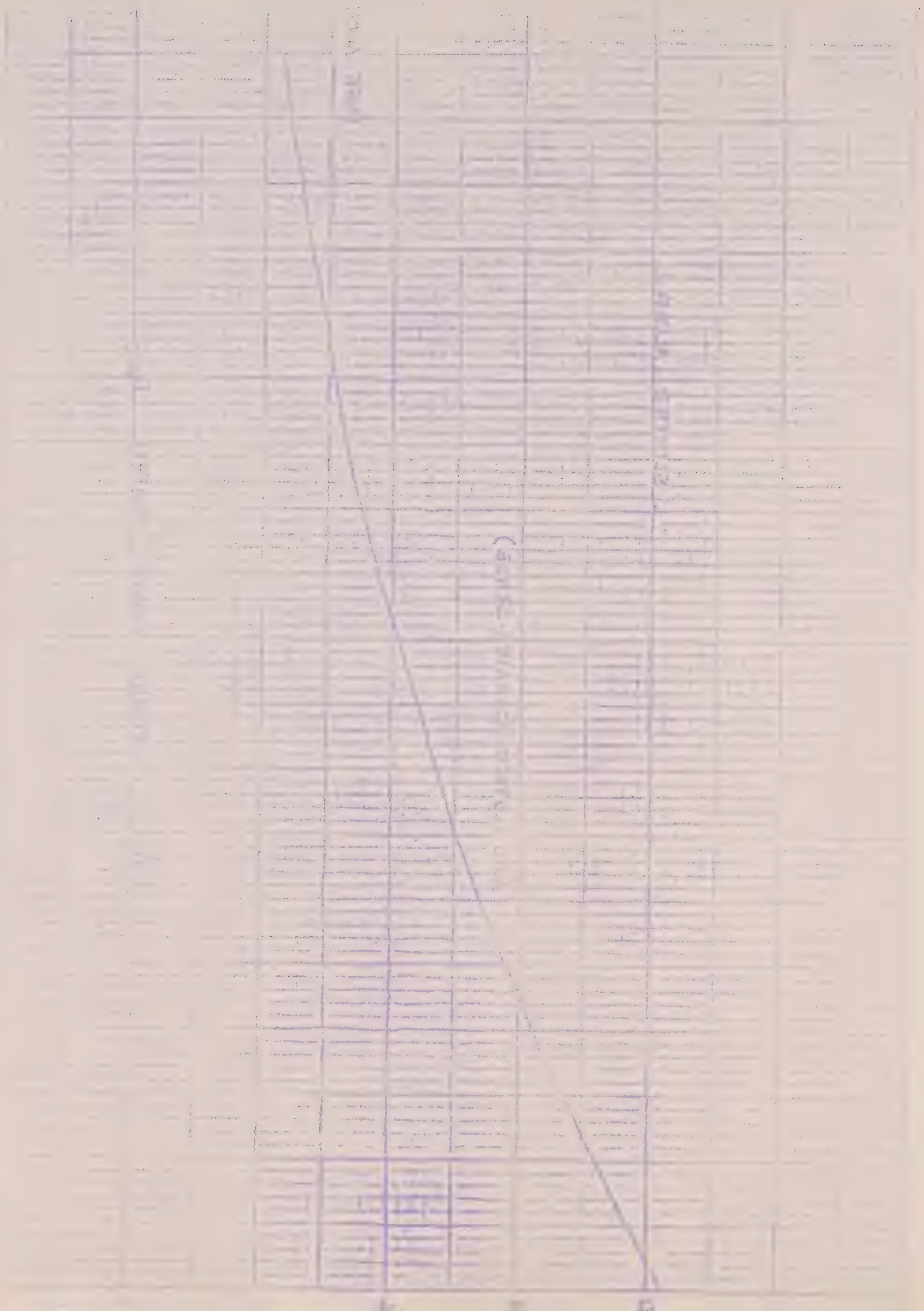


fig 3 OPTIMUM CHANNEL TRAFFIC G_{opt}



APPENDIX

EXACT ANALYSIS

A.1 Distribution of y $F_y(x)$

Knowing that n packets arrived in a sec, the arrival times $x_1, x_2, \ldots, x_n$ of the n packets are independent random variables uniformly distributed over $[0,a]$.

$$\Pr\{y \leq x \mid n\} = \Pr\{\max_{i=0,\ldots,n} x_i \leq x\}$$

$$= \left(\frac{x}{a}\right)^n$$

$$F_y(x) = \Pr\{y \leq x\} = \sum_{n=0}^{\infty} \left(\frac{x}{a}\right)^n \frac{(aG)^n}{n!} e^{-aG}$$

$$= e^{-aG} e^{Gx} \qquad 0 \leq x \leq a$$

The average is given by $\bar{y} = a - \frac{1}{G}(1 - e^{-aG})$ and the Laplace transform of the density function is

$$F_y^*(s) = e^{-Ga} + \frac{G[e^{-sa} - e^{-Ga}]}{G - s}$$

A.2 Distribution of Number of Packets Accumulated at End of A Transmission Period

Let $q_n(x) = \Pr\{n \text{ packets accumulated at end of transmission period} \mid y = x\}$

and $$q_n = \int_{x=0}^{a} q_n(x)\,dF_y(x)$$

The generating function of the latter is

$$Q(z) = e^{G(z-1)} \left[e^{-Ga} + \frac{G[e^{aG(z-1)} - e^{-aG}]}{G - G(1-z)} \right]$$

$$= e^{-G(1-z)} e^{-Ga} \left[1 + \frac{e^{aGz} - 1}{z} \right]$$

Therefore

$$q_0 = Q(z)\Big|_{z=0} = e^{-G(1+a)}[1 + aG]$$

and

$$q_1 = e^{-G(1+a)} \left[G(1 + aG) + \frac{a^2G^2}{2} \right]$$

For small a

$$q_1 \simeq G(1 + aG)e^{-G(1+a)}$$

and $$\frac{q_1}{q_0} = G$$

A.3 Average Busy Period

Let y_k denote the random variable y defined above corresponding to the kth transmission period in a busy period. All y_k, $k = 1,2,\ldots,$ are independent and identically distributed.

Conditioned on the fact we know the sequence $\{y_k = x_k\}$ for all $k \geq 1$, the average busy period is

$$\bar{B}(x_1,x_2,\ldots) = \sum_{k=1}^{\infty} [k(1+a) + x_1 + x_2 + \ldots + x_k]\, q_0(x_k) \prod_{i=1}^{k-1} (1 - q_0(x_i))$$

Therefore, removing the condition:

$$\bar{B} = \ldots \int_{x_i=0}^{a} \ldots \int_{x_i=0}^{a} \bar{B}(x_1,x_2,\ldots)\, dF_{y_1}(x_1)\, dF_{y_2}(x_2) \ldots$$

It is easy to see that the contribution of the term $k(1 + a)$ is

$$\sum_{k=1}^{\infty} \int_{x_1=0}^{a} \ldots \int_{x_k=0}^{a} k(1+a) q_0(x_k) \prod_{i=1}^{k-1} (1 - q_0(x_i))\, dF_{y_1}(x_1) \ldots dF_{y_k}(x_k)$$

$$= \sum_{k=1}^{\infty} k(1+a)\, q_0 (1 - q_0)^{k-1} = \frac{1+a}{q_0}$$

The contribution of the term x_k is

$$\sum_{j=k}^{\infty} \int_{x_1=0}^{a} \cdots \int_{x_j=0}^{a} x_k\, q_0(x_j) \prod_{i=1}^{j-1} (1 - q_0(x_i))\, dF_{y_1}(x_1) \cdots dF_{y_j}(x_j)$$

$$= \int_{x_k=0}^{a} x_k\, q_0(x_k)(1 - q_0)^{k-1} dF_{y_k}(x_k) + \sum_{j=k+1}^{\infty} \int_{x_k=0}^{a} x_k\, q_0(1 - q_0)^{j-2}(1 - q_0(x_k))\, dF_{y_k}(x_k)$$

$$= (1 - q_0)^{k-1} \int_{x_k=0}^{a} x_k q_0(x_k)\, dF_{y_k}(x_k) + \left[\int_{x_k=0}^{a} x_k(1 - q_0(x_k))\, dF_{y_k}(x_k)\right] \sum_{j=k+1} q_0(1 - q_0)^{j-2}$$

$$= \bar{y}(1 - q_0)^{k-1}$$

Finally

$$\bar{B} = \frac{1 + a}{q_0} + \sum_{k=1}^{\infty} \bar{y}(1 - q_0)^{k-1} = \frac{1 + a + \bar{y}}{q_0}$$

Probability of Success

Following the same approach as in Section IV.2, we have

$$P_r\{\text{success}\} = \frac{\frac{1 + \bar{y}}{q_0} e^{-Ga} q_0 + \frac{1}{G} e^{-Ga}}{\frac{1 + a + \bar{y}}{q_0} + \frac{1}{G}}$$

$$= \frac{[1 + G(1 + \bar{y})](1 + aG)\ e^{-G(1+2a)}}{G(1 + a + \bar{y}) + (1 + aG)\ e^{-(1+a)G}}$$

$$\bar{y} = a - \frac{1}{G}(1 - e^{-aG}) \quad \downarrow$$

$$= \frac{(1 + aG)(G + aG + e^{-aG})\ e^{-G(1+2a)}}{G + 2aG - (1 - e^{-aG}) + (1 + aG)\ e^{-G(1+a)}}$$